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$\mathcal{H}_\infty$ state-feedback control for fuzzy systems with input saturation via fuzzy weighting-dependent Lyapunov functions

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ABSTRACT

This paper proposes a method for designing an $\mathcal{H}_\infty$ state-feedback fuzzy controller for discrete-time Takagi–Sugeno (T–S) fuzzy systems with input saturation. To address the input saturation problem, this paper first formulates a set invariance condition for the T–S fuzzy systems. Then, based on the set invariance condition, this paper establishes $\mathcal{H}_\infty$ stabilization conditions associated with a fuzzy weighting-dependent Lyapunov function, where the fuzzy controller is designed to be dependent on not only the current-time but also the one-step-past information on the time-varying fuzzy weighting functions. In the derivation, the $\mathcal{H}_\infty$ stabilization conditions are first formulated in terms of parameterized linear matrix inequalities (PLMIs), and then reconverted into LMIs by an efficient relaxation technique.

1. Introduction

Over the past two decades, there has been rapidly growing interest in approximating the nonlinear system by a Takagi–Sugeno (T–S) fuzzy mode, see [1–3] and the references therein. Besides, based on the T–S fuzzy model, one has developed various model-based fuzzy controls stabilizing the nonlinear systems. Generally, there are two approaches in designing the model-based fuzzy control. One is based on the common quadratic Lyapunov function (CQLF) [4,5], and the other is based on the Lyapunov function associated with the fuzzy weighting functions, such as piecewise Lyapunov function (PLF) [6,7] and fuzzy weighting-dependent Lyapunov function (FWDLF) [8,9]. For the reason that the former approach leads to conservative results, recent research efforts have mainly focused on using the latter one in designing a fuzzy controller. Thus, this paper also intends to use the latter approach based on the FWDLF of mapping from fuzzy weighting functions to a Lyapunov matrix.

In addition, this paper handles the input saturation problem occurred frequently in many practical control applications. Since the existence of saturation is often the source of instability in control systems, numerous investigation and research efforts are already underway to deal with the input saturation problem, see [10,11] and the references therein. In [10], Cao and Lin employed the fuzzy control approach to deal with the stability and stabilization problems of nonlinear systems with actuator saturation, where they identified a set invariance condition to directly incorporate the actuator saturation. After that, in [11], Lee et al. proposed an analysis and design methodology for robust control of affine-in-control nonlinear systems subject to actuator saturation, based on the introduction of the fuzzy Kronecker delta. However, one thing that leaves something to be desired is that they commonly used the approach based on the CQLF.

Thus, this paper will design an $\mathcal{H}_\infty$ fuzzy control system based on the FWDLF approach, so that the $\mathcal{L}_2$ gain of the mapping from the exogenous disturbance to the desired output is minimized or no larger than some prescribed value. Methodologically, this paper first formulates a set invariance condition for the T–S fuzzy systems. Then, based on the

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set invariance condition, this paper establishes $\mathcal{H}_\infty$ stabilization conditions associated with the FWDLF, where the fuzzy controller is designed to be dependent on not only the current-time but also the one-step-past information on the time-varying fuzzy weighting functions. In the derivation, the $\mathcal{H}_\infty$ stabilization conditions are first formulated in terms of parameterized linear matrix inequalities (PLMIs), and then reconverted into LMIs by an efficient relaxation technique. Finally, from the solutions of the LMIs, this paper reconstructs an $\mathcal{H}_\infty$ fuzzy controller by using the so-called non-parallel distributed compensation (non-PDC) scheme.

The paper is organized as follows. Section 2 gives a mathematical description of a discrete-time T–S fuzzy system. Section 3 formulates the conditions for $\mathcal{H}_\infty$ stabilization in terms of PLMIs. Furthermore, Section 4 presents an LMI-based relaxed version of the proposed $\mathcal{H}_\infty$ stabilization condition. Section 5 demonstrates the performance of the relaxed $\mathcal{H}_\infty$ fuzzy control. Finally, in Section 6, the concluding remarks are made.

Notation: Notations in this paper are fairly standard. For $x \in \mathcal{R}^n$, $\|x\|$ is taken to be the standard Euclidian norm., *i.e.*, $\|x\| = (x^{\mathrm{T}}x)^{1/2}$. And, inequalities between vectors mean componentwise inequalities. Besides, in symmetric block matrices, $(*)$ is used as an ellipsis for terms that are induced by symmetry.

2. Problem statement

Consider a compact discrete-time T–S fuzzy system of the following form:

$$\begin{aligned} x_{k+1} &= A(\Theta_k)x_k + B_1(\Theta_k)w_k + B_2(\Theta_k)\ \mathrm{sat}(u_k, \bar{u}), \\ z_k &= C(\Theta_k)x_k + D_1(\Theta_k)w_k + D_2(\Theta_k)\ \mathrm{sat}(u_k, \bar{u}), \end{aligned} \tag{1}$$

subject to

$$\begin{bmatrix} A(\Theta_k) & B_1(\Theta_k) & B_2(\Theta_k) \\ C(\Theta_k) & D_1(\Theta_k) & D_2(\Theta_k) \end{bmatrix} \stackrel{\triangle}{=} \sum_{i=1}^{r} \theta_i(\eta_k) \begin{bmatrix} A_i & B_{1i} & B_{2i} \\ C_i & D_{1i} & D_{2i} \end{bmatrix}, \tag{2}$$

where $x_k \in \mathcal{R}^{n_x}$, $u_k \in \mathcal{R}^{n_u}$, $w_k \in \mathcal{R}^{n_w}$ and $z_k \in \mathcal{R}^{n_z}$ denote the state, the input, the disturbance and the performance output, respectively; r denotes the number of system rules; $\eta_k \in \mathcal{R}^p$ denotes the premise variable vector that may depend on the state in many cases; $\theta_i(\eta_k)$ denote normalized time-varying fuzzy weighting functions for each rule at time k; and $\Theta_k \in \mathcal{R}^r$ stands for a vector of time-varying fuzzy weighting functions $\theta_i(\eta_k)$ at time k. In (1), the saturation function $\mathrm{sat}(u, \bar{u})$ means

$$\mathrm{sat}(u, \bar{u}) = [s_1 \cdots s_{n_u}]^{\mathrm{T}}, \qquad s_i = \mathrm{sign}(u_i) \min\{\bar{u}_i, |u_i|\}, \tag{3}$$

where $\bar{u} \in \mathcal{R}^{n_u}$ denotes the saturation level, $\mathrm{sign}(\cdot)$ returns the signs of the corresponding argument, and u_i and $\bar{u}_i$ denote the ith element of $u \in \mathcal{R}^{n_u}$ and $\bar{u} \in \mathcal{R}^{n_u}$, respectively. Throughout this paper, we assume that the disturbance $w_k \in \mathcal{L}_2$ is unknown but belongs to the following set $\mathcal{W}$:

$$\mathcal{W} \stackrel{\triangle}{=} \left\{ w \in \mathcal{R}^{n_w} \mid \|w_k\|^2 \le \bar{w},\ \bar{w} > 0, \forall k \ge 0 \right\}. \tag{4}$$

For effective handling of the saturation nonlinearity (3), we shall use the following polytopic representation method proposed in [12].

Lemma 2.1. *Let $\mathcal{G}$ be the set of $n_u \times n_u$ diagonal matrices whose diagonal elements are 1 or 0. Suppose that $|v_i| \le \bar{u}_i$ for all $i \in [1, n_u]$, where v_i and $\bar{u}_i$ denote the ith element of $v \in \mathcal{R}^{n_u}$ and $\bar{u} \in \mathcal{R}^{n_u}$, respectively. Then*

$$\mathrm{sat}(u, \bar{u}) = \sum_{s=1}^{2^{n_u}} \xi_s \left(G_s u + \bar{G}_s v \right), \quad \sum_{s=1}^{2^{n_u}} \xi_s = 1,\ \xi_s \ge 0, \tag{5}$$

where G_s denote all elements of $\mathcal{G}$, $\bar{G}_s = I - G_s$.

Now, let us consider a state-feedback control law dependent on not only the current-time fuzzy weighting function vector Θ_k but also the one-step-past fuzzy weighting function vector Θ_{k-1} at time k:

$$u_k = F(\Theta_{k-1}, \Theta_k)x_k, \tag{6}$$

$$v_k = H(\Theta_{k-1}, \Theta_k)x_k, \tag{7}$$

where an auxiliary control input v_k is employed to utilize Lemma 2.1 in handling the input saturation (3). Then, the resultant closed-loop system subject to $x_k \in \mathcal{L}(H(\Theta_{k-1}, \Theta_k)) \stackrel{\triangle}{=} \{x \in \mathcal{R}^{n_x} \mid -\bar{u} \le H(\Theta_{k-1}, \Theta_k)x \le \bar{u}\}$, for $k \ge 0$, is given as follows:

$$\begin{aligned} x_{k+1} &= \hat{A}(\Theta_{k-1}, \Theta_k, \Xi_k)x_k + B_1(\Theta_k)w_k, \\ z_k &= \hat{C}(\Theta_{k-1}, \Theta_k, \Xi_k)x_k + D_1(\Theta_k)w_k, \end{aligned} \tag{8}$$

where $\Xi_k \in \mathcal{R}^{2^{n_u}}$ denotes a vector of the interpolation parameters $\xi_s(k)$ in (5),

$$\hat{A}(\cdot) \triangleq A(\Theta_k) + B_2(\Theta_k)\left(G(\Xi_k)F(\Theta_{k-1},\Theta_k) + \bar{G}(\Xi_k)H(\Theta_{k-1},\Theta_k)\right), \tag{9}$$

$$\hat{C}(\cdot) \triangleq C(\Theta_k) + D_2(\Theta_k)\left(G(\Xi_k)F(\Theta_{k-1},\Theta_k) + \bar{G}(\Xi_k)H(\Theta_{k-1},\Theta_k)\right), \tag{10}$$

$$G(\Xi_k) \triangleq \sum_{s=1}^{2^{nu}} \xi_s(k)G_s, \qquad \bar{G}(\Xi_k) \triangleq \sum_{s=1}^{2^{nu}} \xi_s(k)\bar{G}_s. \tag{11}$$

Before ending this section, let us note that the fuzzy weighting functions $\theta_i(\eta_k)$ generally have the following constraints (C1)–(C2) for all time k (see [13,8]):

(C1) $0 \le \alpha_i \le \theta_i(\eta_k) \le \beta_i \le 1, \quad \forall i \in [1, r]$,

(C2) $\sum_{i=1}^{r} \theta_i(\eta_k) = 1$.

3. PLMI-based conditions

3.1. Set invariance condition

In this subsection, we shall derive a condition for obtaining the following ellipsoidal set: $\mathcal{E}(P(\Theta_k)) \triangleq \{x \in \mathcal{R}^{n_x} \mid x^{\mathrm{T}}P(\Theta_k)x \le 1, P(\Theta_k) > 0\}$ such that, for all $k \ge 0$,

$$x_{k+1} \in \mathcal{E}(P(\Theta_k)) \text{ subject to } x_k \in \mathcal{E}(P(\Theta_{k-1})), \quad w_k \in \mathcal{W}. \tag{12}$$

Here, note that if the ellipsoidal set $\mathcal{E}(P(\Theta_{k-1}))$ under the set invariance condition (12) is in the linear region $\mathcal{L}(H(\Theta_{k-1},\Theta_k))$ for all time k, then the transition of the state x_k is always determined by the closed-loop system (8). That is, the condition $\mathcal{E}(P(\Theta_{k-1})) \subset \mathcal{L}(H(\Theta_{k-1},\Theta_k))$ plays an important role for the local stabilization of the discrete-time fuzzy system (1).

Remark 3.1. The matrix $P(\cdot)$ plays a key role in determining the invariant ellipsoidal set $\mathcal{E}(P(\cdot))$ dependent on the fuzzy weighting function, and the matrix $H(\cdot)$ plays in determining the auxiliary control $v(k)$ required in handling the input saturation via Lemma 2.1.

Thus, if $\mathcal{E}(P(\Theta_{k-1})) \subset \mathcal{L}(H(\Theta_{k-1},\Theta_k))$ holds, then the condition $x_{k+1} \in \mathcal{E}(P(\Theta_k))$ in (12) is represented as

$$0 \le \begin{bmatrix} x_k \\ w_k \\ 1 \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} -\hat{A}^{\mathrm{T}}(\cdot)P(\Theta_k)\hat{A}(\cdot) & (*) & 0 \\ -B_1^{\mathrm{T}}(\Theta_k)P(\Theta_k)\hat{A}(\cdot) & -B_1^{\mathrm{T}}(\Theta_k)P(\Theta_k)B_1(\Theta_k) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_k \\ w_k \\ 1 \end{bmatrix}, \tag{13}$$

and furthermore, based on the S-procedure [14], the set invariance condition (12) is converted into

$$0 \le \left[\begin{array}{cc|c} -\hat{A}^{\mathrm{T}}(\cdot)P(\Theta_k)\hat{A}(\cdot) + \rho P(\theta_{k-1}) & (*) & 0 \\ -B_1^{\mathrm{T}}(\Theta_k)P(\Theta_k)\hat{A}(\cdot) & -B_1^{\mathrm{T}}(\Theta_k)P(\Theta_k)B_1(\Theta_k) + \delta I & 0 \\ \hline 0 & 0 & 1-\rho-\delta\bar{w} \end{array}\right], \tag{14}$$

where $\rho > 0$ and $\delta > 0$ correspond to the multiplier coefficients of the S-procedure. Consequently, by (14), we can obtain the following conditions:

$$0 \le \begin{bmatrix} -\hat{A}^{\mathrm{T}}(\cdot)P(\Theta_k)\hat{A}(\cdot) + \rho P(\Theta_{k-1}) & (*) \\ -B_1^{\mathrm{T}}(\Theta_k)P(\Theta_k)\hat{A}(\cdot) & -B_1^{\mathrm{T}}(\Theta_k)P(\Theta_k)B_1(\Theta_k) + \delta I \end{bmatrix}, \tag{15}$$

$$0 < \delta \le (1/\bar{w})(1-\rho). \tag{16}$$

As can be known in (15), for any given $\rho > 0$, the feasibility of (15) always increases as δ increases. Thus, $\delta = (1/\bar{w})(1-\rho)$, and hence, the conditions (15) and (16) become, with the help of Schur complements,

$$0 \le \begin{bmatrix} \rho P(\Theta_{k-1}) & 0 & (*) \\ 0 & \dfrac{(1-\rho)}{\bar{w}}I & (*) \\ \hat{A}(\Theta_{k-1},\Theta_k,\Xi_k) & B_1(\Theta_k) & P^{-1}(\Theta_k) \end{bmatrix}, \tag{17}$$

where $0 < \rho < 1$.

Lemma 3.1. *For a prescribed scalar $0 < \rho < 1$, suppose that there exist $\bar{P}(\Theta_{k-1})$, $\bar{P}(\Theta_k)$, $\bar{F}(\Theta_{k-1},\Theta_k)$, $\bar{H}(\Theta_{k-1},\Theta_k)$, and $\bar{w} > 0$ such that*

$$0 \le \begin{bmatrix} \rho\bar{P}(\Theta_{k-1}) & 0 & (*) \\ 0 & \dfrac{(1-\rho)}{\bar{w}}I & (*) \\ \mathbf{U}_{1s} & B_1(\Theta_k) & \bar{P}(\Theta_k) \end{bmatrix}, \quad \forall s \in [1, 2^{n_u}], \tag{18}$$

$$0 \le \begin{bmatrix} Z & \bar{H}(\Theta_{k-1}, \Theta_k) \\ (*) & \bar{P}(\Theta_{k-1}) \end{bmatrix}, \quad 0 \le \bar{u}_\nu^2 - Z_\nu, \forall \nu \in [1, n_u], \tag{19}$$

where $\bar{P}(\Theta_{k-1}) \triangleq P^{-1}(\Theta_{k-1})$, $\bar{P}(\Theta_k) \triangleq P^{-1}(\Theta_k)$,

$$\mathbf{U}_{1s} \triangleq A(\Theta_k)\bar{P}(\Theta_{k-1}) + B_2(\Theta_k)\left(G_s\bar{F}(\Theta_{k-1}, \Theta_k) + \bar{G}_s\bar{H}(\Theta_{k-1}, \Theta_k)\right).$$

Then the set invariance condition (12) holds for all admissible grades Θ_{k-1}, Θ_k, and disturbances $w_k \in \mathcal{W}$. Moreover, the maximum upper bound $\bar{w}^*$ can be obtained by solving the following optimization problem: for all $s \in [1, 2^{n_u}]$ and $\nu \in [1, n_u]$,

$$\bar{w}^* = \max\ \bar{w} \text{ subject to (18) and (19).} \tag{20}$$

Proof. Pre- and post-multiplying $T_1^{\mathrm{T}} \triangleq \mathrm{diag}(\bar{P}(\Theta_{k-1}), I, I)$ and T_1 on the right-hand side of (17) yields

$$0 \le \begin{bmatrix} \rho\bar{P}(\Theta_{k-1}) & 0 & (*) \\ 0 & \dfrac{(1-\rho)}{\bar{w}}I & (*) \\ \hat{A}(\Theta_{k-1}, \Theta_k, \Xi_k)\bar{P}(\Theta_{k-1}) & B_1(\Theta_k) & \bar{P}(\Theta_k) \end{bmatrix}, \tag{21}$$

whose (3, 1) block matrix becomes, by (9) and (11),

$$\hat{A}(\Theta_{k-1}, \Theta_k, \Xi_k)\bar{P}(\Theta_{k-1}) = A(\Theta_k)\bar{P}(\Theta_{k-1}) + \sum_{s=1}^{2^{n_u}} \xi_s(k)B_2(\Theta_k)\left(G_s\bar{F}(\Theta_{k-1}, \Theta_k) + \bar{G}_s\bar{H}(\Theta_{k-1}, \Theta_k)\right), \tag{22}$$

where

$$\bar{F}(\Theta_{k-1}, \Theta_k) \triangleq F(\Theta_{k-1}, \Theta_k)\bar{P}(\Theta_{k-1}),$$
$$\bar{H}(\Theta_{k-1}, \Theta_k) \triangleq H(\Theta_{k-1}, \Theta_k)\bar{P}(\Theta_{k-1}).$$

From the fact that multiplying (18) by $\xi_s(k)$ and summing it from $i = 1$ to $i = 2^{n_u}$ yields (21), we can know that, if condition (18) holds for all $s \in [1, 2^{n_u}]$, condition (17) also holds.

Meanwhile, based on [14], the constraint $\mathcal{E}(P(\Theta_{k-1})) \subset \mathcal{L}(H(\Theta_{k-1}, \Theta_k))$ can be converted into

$$0 \le \begin{bmatrix} Z & H(\Theta_{k-1}, \Theta_k) \\ (*) & P(\Theta_{k-1}) \end{bmatrix}, \tag{23}$$

$$0 \le \bar{u}_\nu^2 - Z_\nu, \quad \forall \nu \in [1, n_u], \tag{24}$$

where Z_ν denotes the νth diagonal element of Z. Hence, we can obtain condition (19) by pre- and post-multiplying T_2^{T} and $T_2 = \mathrm{diag}(I, \bar{P}(\Theta_{k-1}))$ on the right-hand side of (23). ■

3.2. $\mathcal{H}_\infty$ stabilization condition

Consider the following FWDLF candidate $V(x_k)$:

$$V(x_k) = x_k^{\mathrm{T}}P(\Theta_{k-1})x_k, \quad P(\Theta_{k-1}) > 0. \tag{25}$$

Based on the FWDLF candidate, the following theorem presents the method of designing an $\mathcal{H}_\infty$ state-feedback fuzzy controller.

Theorem 3.1. *For a prescribed scalar $0 < \rho < 1$, suppose there exist matrices $\bar{P}(\Theta_{k-1})$, $\bar{P}(\Theta_k)$, $\bar{F}(\Theta_{k-1}, \Theta_k)$, $\bar{H}(\Theta_{k-1}, \Theta_k)$, $\bar{w} > 0$, and $\gamma > 0$ such that, for all $s \in [1, 2^{n_u}]$ and $\nu \in [1, n_u]$,*

PLMIs (18), (19),

$$0 < \begin{bmatrix} \bar{P}(\Theta_{k-1}) & 0 & (*) & (*) \\ 0 & \gamma I & (*) & (*) \\ \mathbf{U}_{1s} & B_1(\Theta_k) & \bar{P}(\Theta_k) & 0 \\ \mathbf{U}_{2s} & D_1(\Theta_k) & 0 & \gamma I \end{bmatrix}, \tag{26}$$

where

$$\mathbf{U}_{1s} \triangleq A(\Theta_k)\bar{P}(\Theta_{k-1}) + B_2(\Theta_k)\left(G_s\bar{F}(\Theta_{k-1},\Theta_k) + \bar{G}_s\bar{H}(\Theta_{k-1},\Theta_k)\right),$$

$$\mathbf{U}_{2s} \triangleq C(\Theta_k)\bar{P}(\Theta_{k-1}) + D_2(\Theta_k)\left(G_s\bar{F}(\Theta_{k-1},\Theta_k) + \bar{G}_s\bar{H}(\Theta_{k-1},\Theta_k)\right).$$

Then, closed-loop system (8) is stabilizable with the $\mathcal{H}_\infty$ performance γ for all admissible grades Θ_{k-1} and Θ_k. Moreover, the minimum $\mathcal{H}_\infty$ performance can be obtained by solving the following optimization problem: for all $s \in [1, 2^{n_u}]$ and $\nu \in [1, n_u]$,

$$\min\ \gamma \text{ subject to (18), (19) and (26).}$$

Proof. Recall that, by conditions (18) and (19), the transition of the state x_k is always determined by the closed-loop system (8). Thus, the following two statement are equivalent [8]:

- The closed-loop system (8) is stable with the $\mathcal{H}_\infty$ performance γ.
- There exist $P(\Theta_{k-1})$ and $P(\Theta_k)$, for all admissible grads Θ_{k-1}, Θ_k and Ξ_k, such that

PLMIs (18), (19),

$$0 < \begin{bmatrix} P(\Theta_{k-1}) & 0 & (*) & (*) \\ 0 & \gamma I & (*) & (*) \\ \hat{A}(\Theta_{k-1},\Theta_k,\Xi_k) & B_1(\Theta_k) & \bar{P}(\Theta_k) & 0 \\ \hat{C}(\Theta_{k-1},\Theta_k,\Xi_k) & D_1(\Theta_k) & 0 & \gamma I \end{bmatrix}. \tag{27}$$

As well known, condition (27) can be derived by

$$V(x_{k+1}) - V(x_k) + z_k^{\mathrm{T}} z_k - \gamma^2 w_k^{\mathrm{T}} w_k < 0. \tag{28}$$

Now, let us pre- and post-multiply T_3^{T} and $T_3 = \mathrm{diag}(\bar{P}(\Theta_{k-1}), I, I, I)$ on the right-hand side of (27). Then

$$0 < \begin{bmatrix} \bar{P}(\Theta_{k-1}) & 0 & (*) & (*) \\ 0 & \gamma I & (*) & (*) \\ (3,1) & B_1(\Theta_k) & \bar{P}(\Theta_k) & 0 \\ (4,1) & D_1(\Theta_k) & 0 & \gamma I \end{bmatrix}, \tag{29}$$

whose $(3, 1)$ and $(4, 1)$ block matrices become, by (9)–(11) and (22) and

$$\hat{C}(\Theta_{k-1},\Theta_k,\Xi_k)\bar{P}(\Theta_{k-1}) = C(\Theta_k)\bar{P}(\Theta_{k-1}) + \sum_{s=1}^{2^{n_u}} \xi_s(k) D_2(\Theta_k)\left(G_s\bar{F}(\Theta_{k-1},\Theta_k) + \bar{G}_s\bar{H}(\Theta_{k-1},\Theta_k)\right),$$

respectively. Hence, from the fact that multiplying (26) by $\xi_s(k)$ and summing it from $s = 1$ to $s = 2^{n_u}$ yield (29), we can know that if condition (26) holds for all $s \in [1, 2^{n_u}]$, then condition (27) also holds. ■

Remark 3.2. The greatest disturbance rejection capability, *i.e.*, the smallest γ, can be obtained by tuning the prescribed scalar ρ between 0 and 1.

4. LMI-based relaxed conditions

Henceforth, for a simple description, we use the following notations: $\theta_i = \theta_i(\eta(k))$ and $\theta_i^- = \theta_i(\eta(k-1))$. Besides, to obtain a finite number of LMIs from the derived PLMIs, we specially select the structures of $\bar{P}(\Theta_{k-1})$, $\bar{P}(\Theta_k)$, $\bar{F}(\Theta_{k-1},\Theta_k)$, and $\bar{H}(\Theta_{k-1},\Theta_k)$ as follows:

$$\bar{P}(\Theta_{k-1}) = \sum_{i=1}^{r} \theta_i^- \bar{P}_i, \qquad \bar{P}(\Theta_k) = \sum_{i=1}^{r} \theta_i \bar{P}_i, \tag{30}$$

$$\bar{F}(\Theta_{k-1},\Theta_k) = \sum_{i=1}^{r} \theta_i^- \bar{F}_{1i} + \sum_{i=1}^{r} \theta_i \bar{F}_{2i}, \tag{31}$$

$$\bar{H}(\Theta_{k-1},\Theta_k) = \sum_{i=1}^{r} \theta_i^- \bar{H}_{1i} + \sum_{i=1}^{r} \theta_i \bar{H}_{2i}. \tag{32}$$

4.1. Relaxation of the set invariance condition

Based on the above structural assumption, condition (18) can be written as follows: for all $s \in [1, 2^{n_n}]$,

$$0 \leq \sum_{\ell=1}^{r} \theta_\ell^- \left(\mathbf{M}_0^{(\ell)} + \sum_{i=1}^{r} \theta_i \left(\mathbf{M}_i^{(s,\ell)} + \left(\mathbf{M}_i^{(s,\ell)} \right)^{\mathrm{T}} \right) + \sum_{i=1}^{r} \theta_i^2 \mathbf{M}_{ii}^{(s)} + \sum_{i=1}^{r} \left(\sum_{j=1}^{i-1} \theta_i \theta_j \mathbf{M}_{ij}^{(s)} + \sum_{j=i+1}^{r} \theta_i \theta_j \left(\mathbf{M}_{ij}^{(s)} \right)^{\mathrm{T}} \right) \right), \tag{33}$$

where $\mathbf{M}_0^{(\ell)} \triangleq \mathrm{diag}(\rho \bar{P}_\ell, \frac{1}{\bar{w}}(1-\rho)I, 0)$,

$$\mathbf{M}_i^{(s,\ell)} \triangleq \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ A_i \bar{P}_\ell + B_{2i}(G_s \bar{F}_{1\ell} + \bar{G}_s \bar{H}_{1\ell}) & B_{1i} & \frac{1}{2}\bar{P}_i \end{bmatrix},$$

$$\mathbf{M}_{ii}^{(s)} \triangleq \begin{bmatrix} 0 & 0 & (*) \\ 0 & 0 & 0 \\ B_{2i}(G_s \bar{F}_{2i} + \bar{G}_s \bar{H}_{2i}) & 0 & 0 \end{bmatrix},$$

$$\mathbf{M}_{ij}^{(s)} \triangleq \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ B_{2i}(G_s \bar{F}_{2j} + \bar{G}_s \bar{H}_{2j}) + B_{2j}(G_s \bar{F}_{2i} + \bar{G}_s \bar{H}_{2i}) & 0 & 0 \end{bmatrix}.$$

Thus, we can easily know that the following condition becomes a sufficient condition of (33): for all $s \in [1, 2^{n_n}]$ and $\ell \in [1, r]$,

$$0 \leq \mathbf{M}_0^{(\ell)} + \sum_{i=1}^{r} \theta_i \left(\mathbf{M}_i^{(s,\ell)} + \left(\mathbf{M}_i^{(s,\ell)} \right)^{\mathrm{T}} \right) + \sum_{i=1}^{r} \theta_i^2 \mathbf{M}_{ii}^{(s)} + \sum_{i=1}^{r} \left(\sum_{j=1}^{i-1} \theta_i \theta_j \mathbf{M}_{ij}^{(s)} + \sum_{j=i+1}^{r} \theta_i \theta_j \left(\mathbf{M}_{ij}^{(s)} \right)^{\mathrm{T}} \right). \tag{34}$$

Now, based on the constraint-elimination methods [14,15], let us convert constraints (C1)–(C2) into, respectively,

$$(\overline{\mathrm{C1}})\ 0 \leq -\sum_{i=1}^{r} \left\{ \theta_i^2 - (\alpha_i + \beta_i)\theta_i + \alpha_i \beta_i \right\} \left(\Lambda_i + \Lambda_i^{\mathrm{T}} \right),$$

$$(\overline{\mathrm{C2}})\ 0 = -\left\{ \sum_{i=1}^{r} \sum_{j=1}^{r} \theta_i \theta_j - 2 \sum_{i=1}^{r} \theta_i + 1 \right\} \left(\Lambda + \Lambda^{\mathrm{T}} \right),$$

where Λ_i and Λ are in $R^{n_c \times n_c}$, $n_c = 2n_x + n_w$, and $0 \leq \Lambda_i + \Lambda_i^{\mathrm{T}}$ for all $i \in [1, r]$. Then, combining constraints $(\overline{\mathrm{C1}})$–$(\overline{\mathrm{C2}})$ becomes

$$0 \leq \mathbf{N}_0 + \sum_{i=1}^{r} \theta_i \left(\mathbf{N}_i + \mathbf{N}_i^{\mathrm{T}} \right) + \sum_{i=1}^{r} \theta_i^2 \mathbf{N}_{ii} + \sum_{i=1}^{r} \left(\sum_{j=1}^{i-1} \theta_i \theta_j \mathbf{N}_{ij} + \sum_{j=i+1}^{r} \theta_i \theta_j \mathbf{N}_{ij}^{\mathrm{T}} \right), \tag{35}$$

where

$$\mathbf{N}_0 \triangleq -\sum_{i=1}^{r} \alpha_i \beta_i \left(\Lambda_i + \Lambda_i^{\mathrm{T}} \right) - (\Lambda + \Lambda^{\mathrm{T}}), \qquad \mathbf{N}_i \triangleq (\alpha_i + \beta_i)\Lambda_i + 2\Lambda,$$

$$\mathbf{N}_{ii} \triangleq -(\Lambda_i + \Lambda_i^{\mathrm{T}}) - (\Lambda + \Lambda^{\mathrm{T}}), \qquad \mathbf{N}_{ij} \triangleq -2\Lambda.$$

The following lemma presents a relaxed condition set for the set invariance.

Lemma 4.1. *For a prescribed scalar $0 < \rho < 1$, suppose that there exist $\bar{P}_i, \bar{F}_{1i}, \bar{F}_{2i}, \bar{H}_{1i}, \bar{H}_{2i}, \Lambda_i$, for $i \in [1, r]$, Λ, Z and $\bar{w} > 0$ such that, for all $s \in [1, 2^{n_u}]$ and $\ell, i, j \in [1, r]$,*

$$0 \leq \mathcal{L}_1 \triangleq \left[\begin{array}{c|ccccc} \Gamma_0^{(\ell)} & (*) & (*) & \cdots & (*) \\ \hline \Gamma_1^{(s,\ell)} & \Phi_1^{(s)} & (*) & \cdots & (*) \\ \Gamma_2^{(s,\ell)} & \Pi_{21}^{(s)} & \Phi_2^{(s)} & \ddots & (*) \\ \vdots & \vdots & \ddots & \ddots & (*) \\ \Gamma_r^{(s,\ell)} & \Pi_{r1}^{(s)} & \cdots & \Pi_{r(r-1)}^{(s)} & \Phi_r^{(s)} \end{array} \right], \tag{36}$$

$$0 \leq \mathcal{L}_2 \triangleq \begin{bmatrix} Z & \bar{H}_{1i} + \bar{H}_{2j} \\ (*) & \bar{P}_i \end{bmatrix}, \qquad Z_\upsilon \leq \bar{u}_\upsilon^2, \quad \forall \upsilon \in [1, n_u], \tag{37}$$

$$0 \leq \Lambda_i + \Lambda_i^{\mathrm{T}}, \tag{38}$$

where $\Gamma_0^{(\ell)} = \mathbf{M}_0^{(\ell)} + \sum_{i=1}^r \alpha_i\beta_i(\Lambda_i + \Lambda_i^{\mathrm{T}}) + (\Lambda + \Lambda^{\mathrm{T}})$, $\Gamma_i^{(s,\ell)} = \mathbf{M}_i^{(s,\ell)} - (\alpha_i + \beta_i)\Lambda_i - 2\Lambda$, $\Phi_i^{(s)} = \mathbf{M}_{ii}^{(s)} + (\Lambda_i + \Lambda_i^{\mathrm{T}}) + (\Lambda + \Lambda^{\mathrm{T}})$, $\Pi_{ij}^{(s)} = \mathbf{M}_{ij}^{(s)} + 2\Lambda$. *Then the set invariance condition* (12) *holds for all admissible grades* Θ_{k-1}, Θ_k, *and disturbances* $w_k \in \mathcal{W}$. *Moreover, the maximum bound* $\bar{w}^*$ *can be obtained by solving the following optimization problem: for all* $s \in [1, 2^{n_u}]$ *and* $\ell, i, j \in [1, r]$,

$$\bar{w}^* = \max\ \bar{w} \text{ subject to (36)–(38).} \tag{39}$$

Proof. By the S-procedure and condition (35), the set invariance condition (34) subject to (C1)–(C2) can be formulated as

$$\begin{aligned} 0 \le \Gamma_0^{(\ell)} &+ \sum_{i=1}^r \theta_i \left(\Gamma_i^{(s,\ell)} + \left(\Gamma_i^{(s,\ell)}\right)^{\mathrm{T}}\right) + \sum_{i=1}^r \theta_i^2 \left(\Phi_i^{(s)} + \left(\Phi_i^{(s)}\right)^{\mathrm{T}}\right) \\ &+ \sum_{i=1}^r \left(\sum_{j=1}^{i-1} \theta_i\theta_j \Pi_{ij}^{(s)} + \sum_{j=i+1}^r \theta_i\theta_j \left(\Pi_{ij}^{(s)}\right)^{\mathrm{T}}\right), \quad \forall s \in [1, 2^{n_u}],\ \ell \in [1, r], \end{aligned} \tag{40}$$

where $\Gamma_0^{(\ell)} = \mathbf{M}_0^{(\ell)} - \mathbf{N}_0$, $\Gamma_i^{(s,\ell)} = \mathbf{M}_i^{(s,\ell)} - \mathbf{N}_i$, $\Phi_i^{(s)} = \mathbf{M}_{ii}^{(s)} - \mathbf{N}_{ii}$, $\Pi_{ij}^{(s)} = \mathbf{M}_{ij}^{(s)} - \mathbf{N}_{ij}$. Furthermore, condition (40) can be rewritten as $0 \le [I\ \theta_1 I\ \cdots\ \theta_r I]\ \mathcal{L}_1\ [I\ \theta_1 I\ \cdots\ \theta_r I]^{\mathrm{T}}$, which can be guaranteed by condition (36).

Meanwhile, since condition (19) is equivalent to $\sum_{i=1}^r \sum_{j=1}^r \theta_i^- \theta_j \mathcal{L}_2$, we can easily know that condition (37) becomes a sufficient condition of (19). Hence, if conditions (36)–(38) hold, the set invariance condition subject to (C1)–(C2) also holds for all possible Θ_k, Θ_{k-1}, and disturbances $w_k \in \mathcal{W}$. ∎

4.2. *Relaxation of the $\mathcal{H}_\infty$ stabilization condition*

Based on the given structural assumptions (30)–(32), condition (26) can be written as follows: for all $s \in [1, 2^{n_u}]$,

$$0 < \sum_{\ell=1}^r \theta_\ell^- \left(\mathbf{R}_0^{(\ell)} + \sum_{i=1}^r \theta_i \left(\mathbf{R}_i^{(s,\ell)} + \left(\mathbf{R}_i^{(s,\ell)}\right)^{\mathrm{T}}\right) + \sum_{i=1}^r \theta_i^2 \mathbf{R}_{ii}^{(s)} + \sum_{i=1}^r \left(\sum_{j=1}^{i-1} \theta_i\theta_j \mathbf{R}_{ij}^{(s)} + \sum_{j=i+1}^r \theta_i\theta_j \left(\mathbf{R}_{ij}^{(s)}\right)^{\mathrm{T}}\right)\right), \tag{41}$$

where $\mathbf{R}_0^{(\ell)} \triangleq \mathrm{diag}(\bar{P}_\ell, \gamma I, 0, \gamma I)$,

$$\mathbf{R}_i^{(s,\ell)} \triangleq \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ A_i\bar{P}_\ell + B_{2i}(G_s\bar{F}_{1\ell} + \bar{G}_s\bar{H}_{1\ell}) & B_{1i} & \frac{1}{2}\bar{P}_i & 0 \\ C_i\bar{P}_\ell + D_{2i}(G_s\bar{F}_{1\ell} + \bar{G}_s\bar{H}_{1\ell}) & D_{1i} & 0 & 0 \end{bmatrix},$$

$$\mathbf{R}_{ii}^{(s)} \triangleq \begin{bmatrix} 0 & 0 & (*) & (*) \\ 0 & 0 & 0 & 0 \\ B_{2i}(G_s\bar{F}_{2i} + \bar{G}_s\bar{H}_{2i}) & 0 & 0 & 0 \\ D_{2i}(G_s\bar{F}_{2i} + \bar{G}_s\bar{H}_{2i}) & 0 & 0 & 0 \end{bmatrix},$$

$$\mathbf{R}_{ij}(s) \triangleq \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ B_{2i}(G_s\bar{F}_{2j} + \bar{G}_s\bar{H}_{2j}) + B_{2j}(G_s\bar{F}_{2i} + \bar{G}_s\bar{H}_{2i}) & 0 & 0 & 0 \\ D_{2i}(G_s\bar{F}_{2j} + \bar{G}_s\bar{H}_{2j}) + D_{2j}(G_s\bar{F}_{2i} + \bar{G}_s\bar{H}_{2i}) & 0 & 0 & 0 \end{bmatrix}.$$

Thus, we can easily know that the following condition becomes a sufficient condition of (41): for all $s \in [1, 2^{n_u}]$ and $\ell \in [1, r]$,

$$0 < \mathbf{R}_0^{(\ell)} + \sum_{i=1}^r \theta_i \left(\mathbf{R}_i^{(s,\ell)} + \left(\mathbf{R}_i^{(s,\ell)}\right)^{\mathrm{T}}\right) + \sum_{i=1}^r \theta_i^2 \mathbf{R}_{ii}^{(s)} + \sum_{i=1}^r \left(\sum_{j=1}^{i-1} \theta_i\theta_j \mathbf{R}_{ij}^{(s)} + \sum_{j=i+1}^r \theta_i\theta_j \left(\mathbf{R}_{ij}^{(s)}\right)^{\mathrm{T}}\right). \tag{42}$$

Now, as in Section 4.1, let us convert the constraints (C1)–(C2) into, respectively,

$$(\underline{\mathrm{C1}})\ 0 \le -\sum_{i=1}^r \left\{\theta_i^2 - (\alpha_i + \beta_i)\theta_i + \alpha_i\beta_i\right\} \left(\Sigma_i + \Sigma_i^{\mathrm{T}}\right),$$

$$(\underline{\mathrm{C2}})\ 0 = -\left\{\sum_{i=1}^r \sum_{j=1}^r \theta_i\theta_j - 2\sum_{i=1}^r \theta_i + 1\right\} \left(\Sigma + \Sigma^{\mathrm{T}}\right),$$

where Σ_i and Σ are in $R^{n_o \times n_o}$, $n_o = 2n_x + n_w + n_z$, and $0 \le \Sigma_i + \Sigma_i^{\mathrm{T}}$, for all $i \in [1, r]$. Then, combining the constraints (C1)–(C2) becomes

$$0 \le \mathbf{S}_0 + \sum_{i=1}^{r} \theta_i \left(\mathbf{S}_i + \mathbf{S}_i^{\mathrm{T}}\right) + \sum_{i=1}^{r} \theta_i^2 \mathbf{S}_{ii} + \sum_{i=1}^{r} \left(\sum_{j=1}^{i-1} \theta_i\theta_j \mathbf{S}_{ij} + \sum_{j=i+1}^{r} \theta_i\theta_j \mathbf{S}_{ij}^{\mathrm{T}} \right), \tag{43}$$

where

$$\mathbf{S}_0 \triangleq -\sum_{i=1}^{r} \alpha_i\beta_i \left(\Sigma_i + \Sigma_i^{\mathrm{T}}\right) - (\Sigma + \Sigma^{\mathrm{T}}), \qquad \mathbf{S}_i \triangleq (\alpha_i + \beta_i)\Sigma_i + 2\Sigma,$$

$$\mathbf{S}_{ii} \triangleq -(\Sigma_i + \Sigma_i^{\mathrm{T}}) - (\Sigma + \Sigma^{\mathrm{T}}), \qquad \mathbf{S}_{ij} \triangleq -2\Sigma.$$

The following theorem presents a relaxed condition set for $\mathcal{H}_\infty$ stabilization under the input constraint.

Theorem 4.1. *For a prescribed scalar $0 < \rho < 1$, suppose that there exist $\bar{P}_i$, $\bar{F}_{1i}$, $\bar{F}_{2i}$, $\bar{H}_{1i}$, $\bar{H}_{2i}$, Λ_i, Σ_i, for $i \in [1, r]$, Λ, Σ, Z, $\bar{w} > 0$ and $\gamma > 0$ such that, for all $s \in [1, 2^{n_u}]$, $\upsilon \in [1, n_u]$, $\ell, i, j \in [1, r]$,*

LMIs (36)–(38),

$$0 < \mathcal{L}_3 \triangleq \left[\begin{array}{c|ccccc} \Upsilon_0^{(\ell)} & (*) & (*) & \cdots & (*) \\ \hline \Upsilon_1^{(s,\ell)} & \Delta_1^{(s)} & (*) & \cdots & (*) \\ \Upsilon_2^{(s,\ell)} & \Psi_{21}^{(s)} & \Delta_2^{(s)} & \ddots & (*) \\ \vdots & \vdots & \ddots & \ddots & (*) \\ \Upsilon_r^{(s,\ell)} & \Psi_{r1}^{(s)} & \cdots & \Psi_{r(r-1)}^{(s)} & \Delta_r^{(s)} \end{array}\right], \tag{44}$$

$$0 \le \Sigma_i + \Sigma_i^{\mathrm{T}}, \tag{45}$$

where $\Upsilon_0^{(\ell)} = \mathbf{R}_0^{(\ell)} + \sum_{i=1}^{r} \alpha_i\beta_i(\Sigma_i + \Sigma_i^{\mathrm{T}}) + (\Sigma + \Sigma^{\mathrm{T}})$, $\Upsilon_i^{(s,\ell)} = \mathbf{R}_i^{(s,\ell)} - (\alpha_i + \beta_i)\Sigma_i - 2\Sigma$, $\Delta_i^{(s)} = \mathbf{R}_i^{(s)} + (\Sigma_i + \Sigma_i^{\mathrm{T}}) + (\Sigma + \Sigma^{\mathrm{T}})$, $\Psi_{ij}^{(s)} = \mathbf{R}_{ij}^{(s)} + 2\Sigma$. Then, the closed-loop system (8) is stable with $\mathcal{H}_\infty$ performance γ for all admissible grades Θ_k and Θ_{k-1}. Moreover, the minimum $\mathcal{H}_\infty$ performance can be obtained by solving the following optimization problem: for all $s \in [1, 2^{n_u}]$, $\upsilon \in [1, n_u]$, $i, j, \ell \in [1, r]$,

$$\min \ \gamma \ \text{subject to (36)–(38), (44) and (45)}. \tag{46}$$

Consequently, the corresponding control gain $F(\Theta_{k-1}, \Theta_k)$ is derived by

$$F(\Theta_{k-1}, \Theta_k) = \bar{F}(\Theta_{k-1}, \Theta_k)\bar{P}^{-1}(\Theta_{k-1}), \tag{47}$$

where $\bar{P}(\Theta_{k-1})$ and $\bar{F}(\Theta_{k-1}, \Theta_k)$ are recovered (30) and (31), respectively.

Proof. By the S-procedure and condition (43), the $\mathcal{H}_\infty$ stabilization condition (42) subject to (C1)–(C2) can be formulated as

$$\begin{aligned} 0 \le\ & \Upsilon_0^{(\ell)} + \sum_{i=1}^{r} \theta_i \left(\Upsilon_i^{(s,\ell)} + \left(\Upsilon_i^{(s,\ell)}\right)^{\mathrm{T}} \right) + \sum_{i=1}^{r} \theta_i^2 \left(\Delta_i^{(s)} - \left(\Delta_i^{(s)}\right)^{\mathrm{T}} \right) \\ & + \sum_{i=1}^{r} \left(\sum_{j=1}^{i-1} \theta_i\theta_j \Psi_{ij}^{(s)} + \sum_{j=i+1}^{r} \theta_i\theta_j \left(\Psi_{ij}^{(s)}\right)^{\mathrm{T}} \right), \quad \forall \ell \in [1, r], \end{aligned} \tag{48}$$

where $\Upsilon_0^{(\ell)} = \mathbf{R}_0^{(\ell)} - \mathbf{S}_0$, $\Upsilon_i^{(s,\ell)} = \mathbf{R}_i^{(s,\ell)} - \mathbf{S}_i$, $\Delta_i^{(s)} = \mathbf{R}_{ii}^{(s)} - \mathbf{S}_{ii}$, $\Psi_{ij}^{(s)} = \mathbf{R}_{ij}^{(s)} - \mathbf{S}_{ij}$. Furthermore, condition (48) can be rewritten as $0 < [I\ \theta_1 I\ \cdots\ \theta_r I]\ \mathcal{L}_3\ [I\ \theta_1 I\ \cdots\ \theta_r I]^{\mathrm{T}}$, which can be guaranteed by condition (44). ■

Remark 4.1. The main contribution of this paper is in establishing a less conservative $\mathcal{H}_\infty$ stabilization condition for input-saturated fuzzy systems by enhancing the interactions among the fuzzy subsystems. In order to search for the optimal solution in Lemma 4.1 and Theorem 4.1, however, we need to vary the variable ρ from 0 to 1.

5. Numerical example

To verify the effective of the proposed fuzzy control law, we consider the problem of balancing and swing-up of an inverted pendulum on a cart (refer [16]). Here, the nonlinear plant (see Fig. 1) is represented by two T–S fuzzy rules ($r = 2$), and linearized around 80° because the plant is not controllable for $x_1 = \pm\pi/2$. Furthermore, to obtain a discrete-time

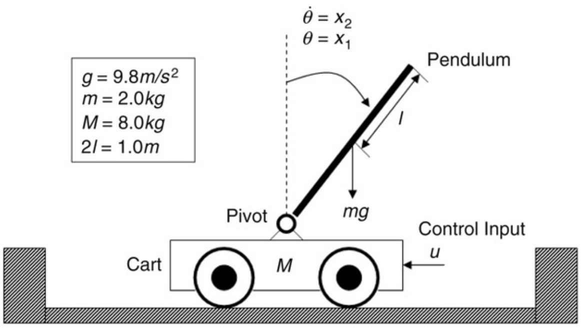

Fig. 1. Inverted-pendulum system.

Table 1
Disturbance tolerance $\bar{w}^*$ according to the saturation level $\bar{u}$ ($\rho = 0.53$).

$\bar{u}$	5	7	10	15
$(\bar{w}^*)^{1/2}$ (Lemma 4.1)	0.1847	0.2587	0.3693	0.5535

Table 2
Minimized $\mathcal{H}_\infty$-performance according to $\bar{u}$ ($\rho = 0.53$, $\bar{w}^{1/2} = 0.1$).

$\bar{u}$	5	7	10	15	No sat.
γ (Theorem 4.1)	9.3987	5.1795	3.1708	2.8478	2.6149

inverted pendulum fuzzy system, we use the bilinear transformation (or called Tustin's transformation) with sampling time $T_s = 0.1$, as in [8]:

$$A_1 = \begin{bmatrix} 1.0904 & 0.1045 \\ 1.8076 & 1.0904 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1.0543 & 0.1027 \\ 1.0863 & 1.0543 \end{bmatrix},$$

$$B_{11} = \begin{bmatrix} 0.0331 \\ 0.0289 \end{bmatrix}, \quad B_{12} = \begin{bmatrix} -0.0338 \\ -0.0441 \end{bmatrix}, \quad B_{21} = \begin{bmatrix} -0.0029 \\ -0.0583 \end{bmatrix}, \quad B_{22} = \begin{bmatrix} -0.0004 \\ -0.0085 \end{bmatrix},$$

$$D_{11} = \begin{bmatrix} 0.0005 \\ 0.1000 \end{bmatrix}, \quad D_{12} = \begin{bmatrix} -0.0006 \\ -0.1000 \end{bmatrix}, \quad D_{21} = \begin{bmatrix} 0.1000 \\ 0.0500 \end{bmatrix}, \quad D_{22} = \begin{bmatrix} -0.1000 \\ -0.0500 \end{bmatrix},$$

$$C_1 = \begin{bmatrix} 0.0331 & 0.0017 \\ 0.0000 & 0.0000 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 0.0351 & 0.0018 \\ 0.0000 & 0.0000 \end{bmatrix}.$$

Based on the premise variable $\eta_k = x_{1k}$, the fuzzy weighting functions $\theta_1(\eta_k)$ and $\theta_2(\eta_k)$ are given as $\theta_1(\eta_k) = 1 - (2/\pi)|\eta_k|$ and $\theta_2(\eta_k) = (2/\pi)|\eta_k|$, respectively. Set $\alpha_1 = \alpha_2 = 0$ and $\beta_1 = \beta_2 = 1$. For $\rho = 0.53$, Table 1 shows the disturbance tolerance $\bar{w}^*$ for the respective saturation level $\bar{u}$, obtained by Lemma 4.1. Moreover, for $\bar{w}^{1/2} = 0.1$ and $\rho = 0.53$, Table 2 shows the minimized $\mathcal{H}_\infty$ performance for the respective saturation level $\bar{u}$, obtained by Theorem 4.1. Here, we simulate the behaviors of the closed-loop systems with $\bar{u} = 5$ and $w_k = 0.2\,\text{rand}(\,) - 0.1$, where rand( ) denotes a random number generator with uniform distribution. Fig. 2(a) shows the saturated control input profiles, and Fig. 2(b),(c) shows the state profiles, where $x_{1,k}$ and $x_{2,k}$ denote the first and second element of the state x_k, respectively.

6. Concluding remarks

For discrete-time Takagi–Sugeno (T–S) fuzzy systems with input saturation, we established an LMI-based relaxed $\mathcal{H}_\infty$ state-feedback fuzzy control law associated with an FWDLF, where the controller is designed by non-PDC scheme. Through a numerical example based on the inverted-pendulum system, we verified in detail the performance of the proposed result.

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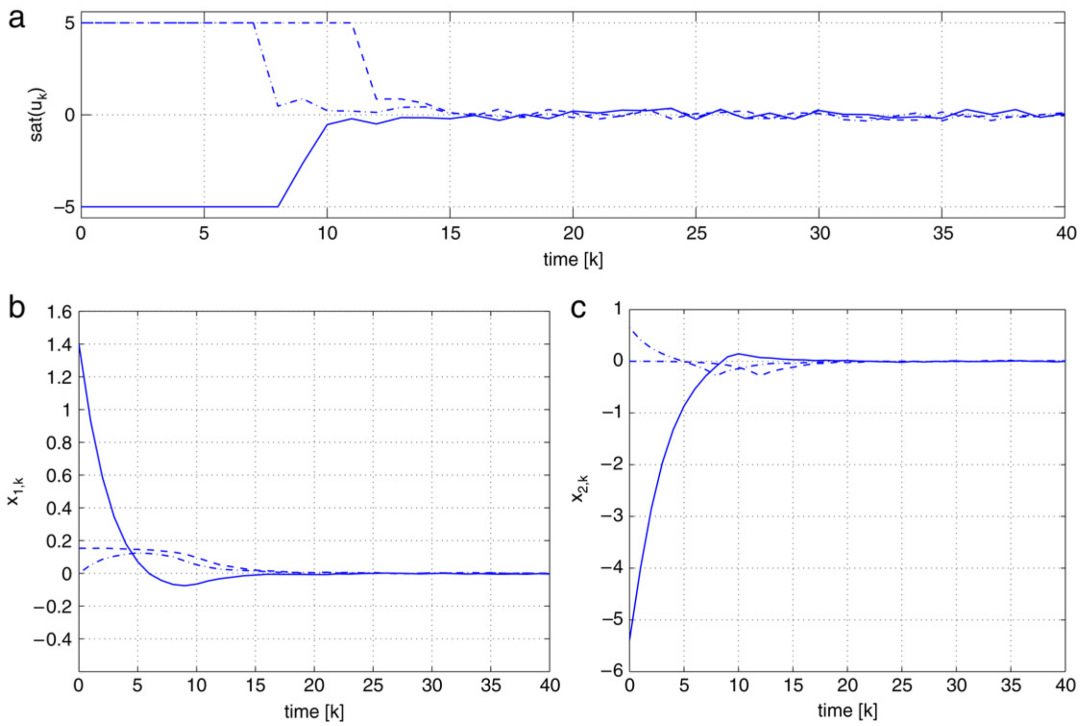

Fig. 2. (a) Saturated control inputs $\mathrm{sat}(u(k))$; (b)–(c) state responds $x_{1,k}$ and $x_{2,k}$ for initial states $x_0 = [1.3963\ -5.39]^{\mathrm{T}}$ (solid-line), $x_0 = [0.0\ 0.647]^{\mathrm{T}}$ (dash-dot-line), $x_0 = [0.153\ 0.0]^{\mathrm{T}}$ (dash-line).

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